Maxwell-Cattaneo's equation and the stability of fluctuations in the relativistic fluid

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Abstract

Extended theories are widely used in the literature to describe the relativistic fluid. The motivation for this is mostly due to the causality issues allegedly present in the first order theories. However, the decay of fluctuations in the system is also at stake when first order theories *that couple heat with acceleration* are used. In this paper it is shown how the generic instabilities in relativistic fluids are not present when a Maxwell-Cattaneo type law is introduced in the system of hydrodynamic equations. Emphasis is made on the fact that the stabilization is only due to the difference in characteristic times for heat flux relaxation and instabilities onset. This gives further evidence that Eckart's like constitutive equations are responsible for the first order sytem exhibiting unphysical behavior.

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I. INTRODUCTION

The transport equations for the relativistic fluid have been well established within the framework of irreversible thermodynamics [1, 2]. However, it was shown that Eckart's constitutive equation, as a closure equation for the heat flux including an acceleration term as driving force [3], renders the system unstable [4] and seems to allow propagation of signals at speeds larger than the speed of light [5]. This instability and acausality in the relativistic gas lead to the formulation and use of higher order or extended theories [6–9].

The definite proof of the violation of Onsager's regression of fluctuations hypothesis was firstly shown by Hiscock and Lindblom eventhough they interpreted this contradiction with the tenets of irreversible thermodynamics as an early onset of instabilities [4]. In their paper it is shown that fluctuations in the linearized system of equations grow exponentially with a very small characteristic time. Meanwhile, buried deep in the fundamental hypothesis of non-equilibrium thermodynamics, Onsager's regression assumption assert that spontaneous fluctuations of microscopic origin should decay following the linearized equations for the state variables. It is this assumption that is violated when Eckart's equation for the heat flux is used, as it is shown in Ref. [10].

As a result, Hiscock and Lindblom suggested in their work that, due to the unphysical behavior predicted by Eckart's formalism, "standard" first order theories should be discarded in favor of the second order one developed by Israel. Even though the causality in the extended theories was at the time verified, to the authors' knowledge the stability within such framework was not assessed.

In this work we revisit the stability analisis for fluctuations within the linearized system by using a Maxwell-Cattaneo type constitutive equation for heat. This calculation not only validates the suggestion in Ref. [4] but also sheds light on the "stabilization" mechanism by showing that the dynamics of the fluctuations are dominated by two competing drives: Eckart's constitutive equation and Cattaneo's damping.

The rest of this paper is divided as follows. In Sect. II the system of transport equations is established and the Maxwell-Cattaneo law for the heat flux is introduced. The equations are linearized for fluctuations around equilibrium values for the state variables in Sect. III. The dispersion relation and its cualitative analysis are shown in Sect. IV, while Sect. V includes the corresponding discussion and final remarks.

II. RELATIVISTIC TRANSPORT EQUATIONS

The transport equations for a simple relativistic fluid are obtained from the conservation equations for the particle and momentum-energy fluxes, that is

$$N^{\mu}_{;\mu} = 0 \tag{1}$$

and

$$T^{\mu\nu}_{;\nu} = 0 \tag{2}$$

where N^{μ} is the particle four flux given by

$$N^{\mu} = n u^{\mu} \tag{3}$$

and

$$T^{\mu}_{\nu} = \frac{n\varepsilon}{c^2} u^{\mu} u_{\nu} + p h^{\mu}_{\nu} + \Pi^{\mu}_{\nu} + \frac{1}{c^2} q^{\mu} u_{\nu} + \frac{1}{c^2} u^{\mu} q_{\nu} \tag{4}$$

is the stress-energy tensor. Here n is the particle number density, u^{μ} the hydrodynamic four-velocity, c the speed of light, ε the internal energy density per particle and p the hydrostatic pressure. The dissipative fluxes are the heat flux q^{ν} and the Navier tensor Π^{μ}_{ν}. The spatial projector h^{μ}_{ν} for a $+++-$ signature is given by

$$h^{\mu}_{\nu} = \delta^{\mu}_{\nu} + \frac{u^{\mu} u_{\nu}}{c^2} \tag{5}$$

which satisfies $h^{\mu}_{\nu} u^{\nu} = 0$ for $\mu = 1, ..., 4$, and the orthogonality conditions implied in this 3+1 representation are

$$u_{\mu} \Pi^{\mu}_{\nu} = 0, \qquad q_{\mu} u^{\mu} = 0 \tag{6}$$

The introduction of Eq. (3) in Eq. (1) yields the relativistic continuity equation, i. e.

$$\dot{n} + n\theta = 0 \tag{7}$$

with $\theta = u^{\nu}_{;\nu}$. Both energy and momentum balances are extracted from Eq. (2). For $\nu = 1, 2, 3$ one obtains the momentum balance

$$\left(\frac{n\varepsilon}{c^2} + \frac{p}{c^2} \right) \dot{u}_{\nu} + \left(\frac{n\dot{\varepsilon}}{c^2} + \frac{p}{c^2}\theta \right) u_{\nu} + p_{,\mu} h^{\mu}_{\nu} + \Pi^{\mu}_{\nu;\mu} + \frac{1}{c^2} \left(q^{\mu}_{;\mu} u_{\nu} + q^{\mu} u_{\nu;\mu} + \theta q_{\nu} + u^{\mu} q_{\nu;\mu} \right) = 0 \tag{8}$$

Considering $\nu = 4$ leads to a total energy balance from which the mechanical energy needs to be substracted in order to establish the heat flux. A shortcut leading directly to the internal energy equation is to consider the projection $u^{\nu} T^{\mu}_{\nu;\mu} = 0$ which leads to

$$n\dot{\varepsilon} + p\theta + u^{\nu}_{,\mu}\Pi^{\mu}_{\nu} + q^{\mu}_{;\mu} + \frac{1}{c^2}\dot{u}^{\nu}q_{\nu} = 0 \tag{9}$$

Using the relation $\dot{\varepsilon} = \left(\frac{\partial\varepsilon}{\partial n}\right)_T \dot{n} + \left(\frac{\partial\varepsilon}{\partial T}\right)_n \dot{T}$ together with the ideal gas law $p = nkT$, which holds in the relativistic case [1], one readily obtains an evolution equation for T, i. e.

$$nC_n\dot{T} + p\theta + u^{\nu}_{;\mu}\Pi^{\mu}_{\nu} + q^{\mu}_{;\mu} + \frac{1}{c^2}\dot{u}^{\nu}q_{\nu} = 0 \tag{10}$$

The set of hydrodynamic equations for the relativistic fluid is thus given by Eqs. (7), (8) and (10). The set is not complete and constitutive equations have to be introduced in order to express the dissipative fluxes in terms of the state variables. In 1940, Eckart proposed, from a purely phenomenological approach by enforcing consistency with the second law of thermodynamics, a coupling of heat with hydrodynamic acceleration additional to the Fourier term [3]. The resulting system of equations was shown to yield unphysical behavior by Hiscock and Lindblom [4]. Following their suggestion, we here study the behavior of the linearized system by introducing for the heat flux, a Maxwell-Cattaneo constitutive relation. Such relation consists in an evolution equation for the heat flux, that is

$$\tau\dot{q}^{\nu} + q^{\nu} = -\kappa h^{\mu\nu}\left(T_{,\nu} + \frac{T}{c^2}\dot{u}_{\nu}\right) \tag{11}$$

where τ is a relaxation time such as $\tau = D_T/C_s$, with C_s the speed of sound. Notice that Eq.(11) becomes Eckart's constitutive equation [3] if $\tau = 0$.

III.   LINEARIZED TRANSPORT EQUATIONS

In this section a set of linearized equations will be obtained by assuming small fluctuations of the state variables around their equilibrium values (denoted by a subscript 0), that is

$$F = F_0 + \delta F \tag{12}$$

Once this substitution is made, second (and higher order) terms on fluctuations are neglected. Then, for the continuity equation we have

$$\delta\dot{n} + n_0\delta\theta = 0 \tag{13}$$

and for the momentum balance

$$\frac{1}{c^2} \left(n_0 \varepsilon_0 + p_0\right) \delta \dot{u}_\nu + kT\delta n_{,\nu} + nk\delta T_{,\nu} - \zeta \delta \theta_{,\nu} - 2\eta \left(\delta \tau_{;\nu}^\mu\right)_{;\mu} + \delta \dot{q}_\nu = 0 \qquad (14)$$

where ζ and η are the bulk and shear viscosity respectively; while the heat equation in this linear approximation reads

$$nC_n \delta \dot{T} + n_0 kT_0 \delta \theta + \delta q_{;\nu}^\nu = 0 \qquad (15)$$

Finally, Maxwell-Cattaneo's equation in terms of fluctuations is given by

$$\tau \delta \dot{q}^\nu + \delta q^\nu = -\kappa h^{\mu\nu} \left(\delta T_{,\nu} + \frac{T_0}{c^2} \delta \dot{u}_\nu\right) \qquad (16)$$

Notice that Eqs. (13) and (15) only depend on the velocity through $\delta\theta$. Because of that, it is convenient to calculate the divergence of Eq. (14) . Following this standard procedure, the longitudinal velocity gradient fluctuation mode is uncoupled and the momentum balance is reduced to a scalar equation for $\delta\theta$

$$\frac{1}{c^2} \left(n_0 \varepsilon_0 + p_0\right) \delta \dot{\theta}_{;\nu} + kT_0 \nabla \delta n + n_0 k \nabla^2 \delta T - \left(\zeta + \frac{4}{3}\eta\right) \nabla^2 \delta\theta + \delta q_{;\nu}^\nu = 0 \qquad (17)$$

Now both the internal energy and momentum linearized balance equations depend on the divergence of the heat flux fluctuations. Thus, if one calculates de divergence of the linearized Maxwell-Cattaneo equation, Eq. (16), the system is reduced to a system of 4 scalar equations for the unknowns δn, $\delta\theta$, δT and $\delta q_{;\nu}^\nu$. The cualitative analysis of such system is carried out in the next section in the Fourier-Laplace space in order to address the behavior of such fluctuations.

IV. THE DISPERSION RELATION

In order to establish the dispersion relation, the system matrix is formed with Laplace and Fourier transforms of equations (13), (15) and (17) and the divergence of Eq. (16), that is:

$$A = \begin{pmatrix} s & n_0 & 0 & 0 \\ -kT_0 q^2 & n_0 ms + q^2 \left(\zeta + \frac{4\eta}{3}\right) & -n_0 kq^2 & \frac{s}{c^2} \\ 0 & n_0 kT_0 & C_n n_0 s & 1 \\ 0 & \frac{sT_0 \kappa}{c^2} & -\kappa q^2 & \tau s + 1 \end{pmatrix}, \qquad (18)$$

with s and q the Laplace and Fourier parameters respectively. The corresponding dispersion relation for the system can thus be written as

$$d_4 s^4 + d_3 s^3 + d_2 s^2 + d_1 s + d_0 = 0 \tag{19}$$

where the coefficients d_0, d_1, d_2 y d_4 are given by:

$$d_0 = \frac{kT_0 \kappa}{C_n m n_0} q^4 \tag{20}$$

$$d_1 = \frac{kT_0}{m} \left[\left(1 + \frac{k}{C_n} \right) q^2 + \frac{1}{n_0} \left(\zeta + \frac{4}{3} \eta \right) \frac{\kappa}{C_n n_0} q^4 \right], \tag{21}$$

$$d_2 = q^2 \left[\frac{1}{mn_0} \left(\zeta + \frac{4}{3} \eta \right) + \frac{\kappa}{C_n n_0} \left(1 - \frac{2kT_0}{mc^2} \right) + \frac{kT_0}{m} \left(1 + \frac{k}{C_n} \right) \tau \right], \tag{22}$$

$$d_3 = 1 + \frac{1}{mn_0} \left(\zeta + \frac{4}{3} \eta \right) \tau q^2, \tag{23}$$

$$d_4 = \tau - \frac{T_0 \kappa}{c^4 m n_0} \tag{24}$$

In order to determine approximate solutions for the fourth order dispersion relation, we follow the same ideas as in Ref. [11]. Since d_4 is a small quantity, three of the roots of Eq. (19) are calculated by neglecting the fourth order term and solving for

$$d_3 s^3 + d_2 s^2 + d_1 s + d_0 = 0 \tag{25}$$

which, using Mountain's method [12], can be shown to yield a real root given by

$$s_1 = -\frac{D_T}{\gamma} \tag{26}$$

and the two imaginary solutions

$$s_{2,3} = -\frac{D_V}{2} - \frac{D_T}{5} - z \left(\frac{5}{6} c^2 \tau - D_T \right) q^2 \pm i \sqrt{\frac{5}{3} \frac{kT}{m}} q \tag{27}$$

where use has been made of the fact that for an ideal gas, for $z << 1$, $C_n = \frac{3}{2} k$, $\gamma = \frac{5}{3}$ and the definitions $D_V = \frac{1}{mn_0} \left(\zeta + \frac{4\eta}{3} \right)$ and $D_T = \frac{\kappa}{n_0 C_n}$ for the viscous and thermal diffusivities have been introduced. These roots give rise to the usual behavior of density fluctuations, that is, a decaying mode with a characteristic time given by s_1 and two oscillating ones with

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frequencies given by the imaginary part of s_2 and s_3. Notice that the only relativistic corrections to this behavior corresponds to the Stokes-Kirchoff coefficient, the term in parenthesis in Eq. (27). Clearly, since the characteristic time τ is of the order of the mean collision time $\sim 10^{-11}s$ and typical thermal diffusivities are in the $10^{-5} - 10^{-7} m^2 s^{-1}$ range, the first term dominates and the relativistic correction yields only a slight increase in the real part of the conjugate roots which vanishes as $z \to 0$ in the non-relativistic limit. It is also important to point out the fact that the relativistic heat diffusion term, zD_T, has opposite sign and, although small, would actually yield a small growth in amplitude.

To approximate the fourth root, we use the information of the previous solutions by assuming they are still approximate roots of the complete equation, which is plausible given the smallness of d_4. Following such proposal, we use the fact that the sum of all roots in an n-th order polynomial is equal to the ratio of the coefficients of the $n-1$ and the n-th power to obtain

$$s_4 = - \left(\tau - \frac{3}{2} \frac{D_T z}{c^2} \right)^{-1} \left[1 + \frac{q^2}{mn_0 c^4} \left(D_V T_0 \kappa + \frac{2}{3} \frac{T_0 \kappa^2}{kn_0} - \frac{4}{3} \frac{T_0^2 \kappa^2}{c^2 mn_0} \right. \right.$$
$$\left. \left. - \frac{2}{3} \frac{c^4 m\kappa\tau}{k} + \frac{4}{3} c^2 T_0 \kappa\tau + \frac{5}{3} \frac{kT_0^2 \kappa\tau}{m} + \frac{5}{3} c^4 kn_0 T_0 \tau^2 \right) \right] \tag{28}$$

Notice that, by taking $\tau = 0$, the fourth root is positive and thus corresponds to a growing mode. In Ref. [11], the existence of this positive large solution was used to argue against Eckart's constitutive equation, in particular the coupling of heat with acceleration. For $\tau \neq 0$, we can approximate

$$s_4 \sim -\frac{1}{\tau} + q^2 \left(D_T + \frac{5}{3} \frac{kT_0}{m} \tau \right) \tag{29}$$

Now, Eq. (29) sheds more light on the issue since, by using the fact that $\tau \gg \frac{T_0 \kappa}{\rho c^4}$, one can conclude not only that the system of hydrodynamic equations with Cattaneo's closure (Eq. (16)) is stable but also realize that this stabilization takes place as the relaxation term dominates over the exponential growth caused by Eckart's coupling. In other words, the time derivative term in Maxwell-Cattaneo's constitutive equations yields not only a modified time scale for the evolution of the fluctuations but, more importantly, changes the overall physical behavior because of the different signs of the terms in the denominator of Eq. (29). The root found in Ref. [11]

$$s_4 \sim \frac{c^4 mn_0}{\kappa T_0} \tag{30}$$

yields an exponential growth in the structure factor while the result in the current calculation

$$s_4 \sim -\frac{1}{\tau} \tag{31}$$

corresponds to a finite spectrum which could eventually be observed and measured.

V.   SUMMARY AND FINAL REMARKS

In the previous section, a dispersion relation for the system of linearized hydrodynamic equations for the relativistic fluid was obtained with a Maxwell-Cattaneo heat evolution equation. Such procedure produced a fourth order equation for the Fourier variable s for a given wavenumber q. This result is similar to the one obtained in Ref. [11] but only in structure since, as was emphatically pointed out in Sect. IV, the overall behavior of the system changes radically. This can be clearly seen by comparing the approximate roots in Eqs. (30) and (31). The former yields an exponential growth in the corresponding structure factor which destroys, theoretically, the spectrum which is clearly unacceptable. However, the latter predicts a decay in time of the fluctuations with a characteristic time determined by τ, the relaxation parameter introduced by Cattaneo.

In this way, the stabilization of the system predicted intuitively by Hiscock and Lindblom in 1985 is confirmed. To our knowledge, only the causality issue had been addressed so far. And moreover, the mechanism for the stabilization can be qualitatively attributed to the competing terms in the denominator of s_4 as given by Eq. (29). By confirming that the use of a relaxation equation for the heat flux eliminates the instability in the system of equation by dominating over the runaway solution that is introduced by the acceleration-heat coupling, verifies the fact that it is such relation the one which in the first place led to the unphysical behavior of thermodynamic fluctuations in the relativistic gas. We thus conclude that the stability of the fluid within Maxwell-Cattaneo's formalism is in complete agreement with the claim that heat cannot be coupled with acceleration.

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[1] C. Cercignani and G. Medeiros Kremer; The Relativistic Boltzmann Equation: Theory and Applications, Cambridge University Press 3rd Ed., UK (1991).

[2] S. R. de Groot, W. A. van Leeuwen and Ch. van der Wert; Relativistic Kinetic Theory, North Holland Publ. Co., Amsterdam (1980).

[3] C. Eckart, Phys. Rev. **58,** 267 (1940); ibid **58,** 919 (1940).

[4] W. A. Hiscock & L. Lindblom, Phys. Rev. D **31**, 725 (1985).

[5] W. Israel, Nonstationary Irreversible Thermodynamics: a Causal Relativistic Theory, Ann. Phys (N.Y.) **100** 310 (1976) and references therein.

[6] W. Israel, Thermodynamics of relativistic systems, Physica **106A** 209 (1981).

[7] W. Israel and J.M. Stewart, progress in relativistic thermodynamics and electrodynamics of continuous media in "General relativity and gravitation" Vol 2; A. Held, editor (Plenum press, N.Y. 1980).

[8] W. Zimdahl, Bulk Viscous Cosmology; Phys. Rev. D **53**, 5483 (1996).

[9] D. Pavon, D. Jou and J. Casas-Vazquez, Ann. Inst. Henri Poincare **A36,** 79 (1982).

[10] A.L Garcia-Perciante, L. S. Garcia-Colin and A. Sandoval-Villalbazo, Gen. Rel. Grav. **41** 1645 (2009).

[11] A.L Garcia-Perciante, L. S. Garcia-Colin and A. Sandoval-Villalbazo, Phys. Rev. E **79**, 066310 (2009).

[12] R. Mountain, Rev. Mod. Phys. **38**, 205 (1966).